Direction-of-Arrival Estimation for Correlated Sources and Low Sample Size

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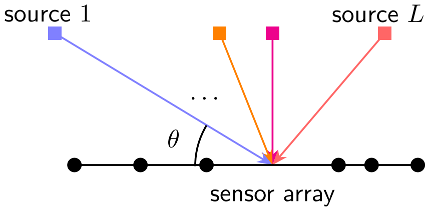
Direction-of-Arrival (DOA) estimation



- Classic problem in e.g. wireless communication, seismic exploration, automatic monitoring
- Numerous approaches:

  maximum likelihood estimator [Stoica, 1989], root-MUSIC [Rao, 1989], ESPRIT [Paulraj, Roy, Kailath, 1989], ...
- Very difficult: correlated sources and low sample size
  - Partial Relaxation (PR) [Trinh-Hoang, 2018]
  - SPARse ROW-norm reconstruction (SPARROW) [Steffens, 2018]

Signal model

- M omnidirectional sensors, narrowband signals located in the farfield of the array
- L source signals with DOAs $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_L]^\mathsf{T}$ (L is known)
- Received signals

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t)$$

  - $\boldsymbol{y}(t) \in \mathbb{C}^M$ received signal vector
  - $\boldsymbol{x}(t) \in \mathbb{C}^L$ source signal vector
  - $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_L)] \in \mathbb{C}^{M \times L}$ steering matrix
  - $\boldsymbol{a}(\theta_i) = [1, e^{-j\pi \sin(\theta_i)}, \ldots, e^{-j(M-1)\pi \sin(\theta_i)}]^\mathsf{T}$ sensor array responses for DOA θ_i with uniform linear array
  - $\boldsymbol{n}(t) \in \mathbb{C}^M$

Signal model

- M omnidirectional sensors, narrowband signals located in the farfield of the array
- L source signals with DOAs $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_L]^\mathsf{T}$ (L is known)
- Multiple snapshots

$$\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{X} + \boldsymbol{N}$$

  - $\boldsymbol{Y} = [\boldsymbol{y}(1), \ldots, \boldsymbol{y}(N)]$: received signal matrix
  - $\boldsymbol{X} = [\boldsymbol{x}(1), \ldots, \boldsymbol{x}(N)]$: source signal matrix
  - $\boldsymbol{N} = [\boldsymbol{n}(1), \ldots, \boldsymbol{n}(N)]$: noise matrix

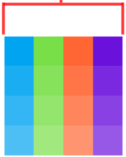
Signal model

- Source signals: stationary process
  - $\mathbb{E}\{\boldsymbol{x}(t)\} = \mathbf{0}$
  - source covariance matrix $\boldsymbol{R}_x = \mathbb{E}\{\boldsymbol{x}(t)\boldsymbol{x}(t)^{\mathsf{H}}\}$
- Noise: i.i.d. white Gaussian
  - $\boldsymbol{R}_{\mathsf{n}} = \mathbb{E}\{\boldsymbol{n}(t)\boldsymbol{n}(t)^{\mathsf{H}}\} = \sigma^2 \boldsymbol{I}_M$
  - σ^2: noise power at each sensor
- Received signals
  - $\boldsymbol{R}_y = \mathbb{E}\{\boldsymbol{y}(t)\boldsymbol{y}(t)^{\mathsf{H}}\} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{R}_x\boldsymbol{A}(\boldsymbol{\theta})^{\mathsf{H}} + \sigma^2 \boldsymbol{I}_M$
  - $\widehat{\boldsymbol{R}}_y = \frac{1}{N}\boldsymbol{Y}\boldsymbol{Y}^{\mathsf{H}}$

Partial Relaxation

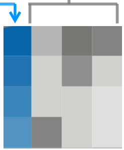
- Conventional

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} f(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{Y})$$



- PR [Trinh-Hoang, 2018]

$$\widehat{\boldsymbol{\theta}} = {}^{L}\arg\min_{\theta} \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(L-1)}} f(\boldsymbol{a}(\theta), \boldsymbol{B}, \boldsymbol{Y})$$



where ${}^{L}\arg\min$: take L deepest minima

  - degrade severely for highly correlated sources or low sample size

SPARROW

- Sparse signal reconstruction

$$\min_{\boldsymbol{Z} \in \mathbb{C}^{K \times N}} \frac{1}{2} ||\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{Z} - \boldsymbol{Y}||_{\mathsf{F}}^2 + \lambda\sqrt{N}||\boldsymbol{Z}||_{2,1} \tag{1}$$

  - $\boldsymbol{\nu} = \{\nu_1, \ldots, \nu_K\}$: sampled FOV with $K \gg L$
  - $\boldsymbol{Z} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K]^{\mathsf{T}}$: row-wise sparse signal matrix
  - $\boldsymbol{A}(\boldsymbol{\nu}) = [\boldsymbol{a}(\nu_1), \ldots, \boldsymbol{a}(\nu_K)] \in \mathbb{C}^{M \times K}$: overcomplete dictionary matrix
  - $\ell_{2,1}$-mixed-norm

$$\|\boldsymbol{Z}\|_{2,1} = \sum_{k=1}^{K} \|\boldsymbol{z}_k\|_2$$

  - $\lambda > 0$: regularization parameter inducing row-sparsity in $\boldsymbol{Z}$
- On-grid assumption: $\theta_i \in \boldsymbol{\nu}$, for $i = 1, \ldots, L$
- Jointly estimating the sources: robust to correlated sources

SPARROW

- Sparse signal reconstruction

$$\min_{\boldsymbol{Z} \in \mathbb{C}^{K \times N}} \frac{1}{2} ||\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{Z} - \boldsymbol{Y}||_{\mathsf{F}}^2 + \lambda\sqrt{N}||\boldsymbol{Z}||_{2,1} \qquad (1)$$

- Problem (1) is equivalent to the convex problem

$$\min_{\boldsymbol{S} \in \mathbb{D}_+^K} \mathrm{Tr}\left((\boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^{\mathsf{H}} + \lambda \boldsymbol{I}_M)^{-1} \widehat{\boldsymbol{R}}_y \right) + \mathrm{Tr}(\boldsymbol{S}) \qquad (2)$$

  - $\mathbb{D}_+^K$: the set of $K \times K$ nonnegative diagonal matrices.

- The solutions of (1) and (2) are related by

$$\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{S}}\boldsymbol{A}(\boldsymbol{\nu})^{\mathsf{H}}(\boldsymbol{A}(\boldsymbol{\nu})\widehat{\boldsymbol{S}}\boldsymbol{A}(\boldsymbol{\nu})^{\mathsf{H}} + \lambda \boldsymbol{I}_M)^{-1}\boldsymbol{Y},$$

$$\widehat{s}_k = \frac{1}{\sqrt{N}}||\widehat{\boldsymbol{z}}_k||_2 \quad \text{for } k = 1, 2, \dots, K,$$

  with $s_1, \dots, s_K \geq 0$ being the diagonal entries of $\boldsymbol{S}$.

Gridless SPARROW

- Uniform linear array with M sensors
  - $\boldsymbol{A}(\boldsymbol{\nu})$ has a Vandemonde structure
  - $\boldsymbol{S}$ nonnegative diagonal
  - positive semidefinite Toeplitz

$$\boldsymbol{T} = \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{\nu})^{\mathsf{H}} = \sum_{k=1}^{K} s_k \boldsymbol{a}(\nu_k)\boldsymbol{a}(\nu_k)^{\mathsf{H}} \in \mathcal{T}_+^M$$

  - $\mathcal{T}_+^M$: the set of $M \times M$ positive semidefinite Toeplitz matrices

- Gridless SPARROW [Steffens, 2018]

$$\widehat{\boldsymbol{T}} = \underset{\boldsymbol{T} \in \mathcal{T}_+^M}{\arg\min} \quad \mathrm{Tr}\left((\boldsymbol{T} + \lambda \boldsymbol{I}_M)^{-1} \widehat{\boldsymbol{R}}_y\right) + \frac{1}{M}\mathrm{Tr}(\boldsymbol{T}) \qquad (3)$$

  - Vandemonde decomposition on $\widehat{\boldsymbol{T}}$ to recover DOAs

Proposed method

- Step 1: solve gridless SPARROW with a reduced regularization parameter

$$\widehat{\boldsymbol{T}} = \underset{\boldsymbol{T}\in\mathcal{T}_+^M}{\arg\min} \quad \mathrm{Tr}\left((\boldsymbol{T}+\lambda\boldsymbol{I}_M)^{-1}\widehat{\boldsymbol{R}}_y\right) + \frac{1}{M}\mathrm{Tr}(\boldsymbol{T})$$

  - $\lambda = C_\lambda \underbrace{\sqrt{\sigma^2 M \log M}}_{\text{empirical}}, \quad C_\lambda = 0.4 \sim 0.6$
  - reduced bias

- Step 2: use PR on $\widehat{\boldsymbol{T}}$ to recover DOAs
  - extract signal subspace from $\widehat{\boldsymbol{T}}$

Simulation results

- Setup:
  - Signal-to-noise ratio: SNR$=1/\sigma^2$
  - Root-mean-square-error:

$$\text{RMSE} = \sqrt{\frac{1}{N_R L} \sum_{i=1}^{N_R} \sum_{l=1}^{L} (\hat{\theta}_l^{(i)} - \theta_l)^2},$$

  - Monte-Carlo runs: 200
- Estimators:
  - The proposed method
  - root-MUSIC
  - PR
  - SPARROW with the original regularization $(C_\lambda = 1)$
  - SPARROW with the same regularization as the proposed method $(C_\lambda = 0.4)$
  - root-MUSIC applied to the Toeplitz matrix solution with regularization $C_\lambda = 0.4$

Simualtion 1: few snapshots

- Two uncorrelated sources at -30° and 30°
- Snapshots $N = 3$, uniform linear array of size $M = 6$

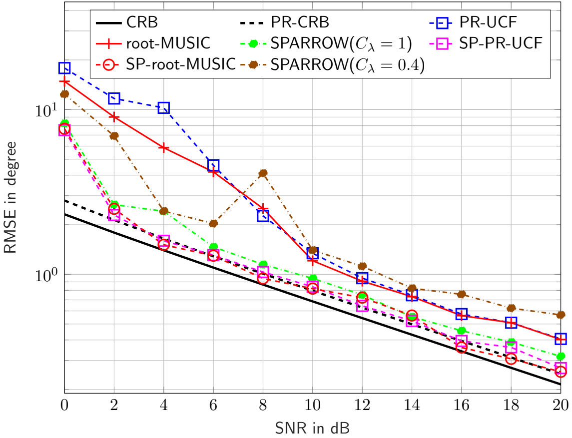


Fig. 1. RMSE vs SNR

Simualtion 2: highly-correlated source signals

- Two sources at 45° and 50°, correlation factor $\rho = 0.95$
- Uniform linear array of size $M = 10$

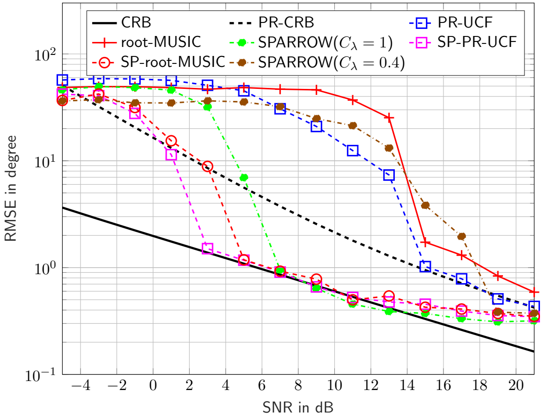


Fig. 2. RMSE vs SNR with snapshots $N = 40$

Simualtion 2: highly-correlated source signals

- Two sources at 45° and 50°, correlation factor $\rho = 0.95$
- Uniform linear array of size $M = 10$

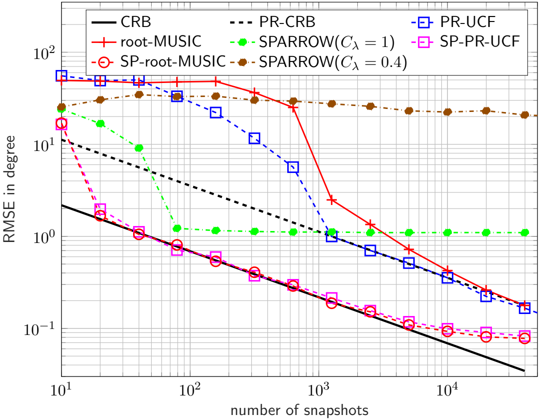


Fig. 3. RMSE vs N with SNR$= 5$ dB

Simualtion 2: highly-correlated source signals

- Two sources at 45° and 50°, correlation factor $\rho = 0.95$
- Uniform linear array of size $M = 10$

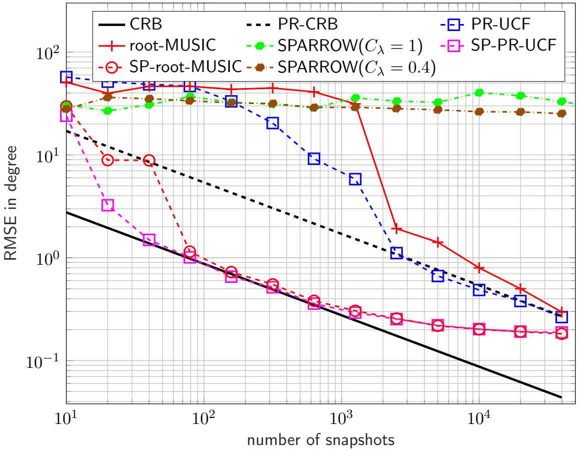


Fig. 4. RMSE vs N with SNR$= 3$ dB

Conclusion

- Take away:
    - Step 1 gridless SPARROW, Step 2 PR
    - robust w.r.t. highly correlated sources and low sample size, superior than both PR and SPARROW
- Generalizes from uniform linear arrays to other array geometries